Problem Set 2 – Relativity (`GOI36A`)

29/09/2020

1 η, δ, ∂, and d

As we have discussed in class, the infinitesimal distance ds between two points in four-dimensional Minkowski spacetime is given by

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{1.1}$$

If we define $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$, then we can more compactly write this infinitesimal distance by

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \ , \tag{1.2}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor, a $(0, 2)$-tensor defined in component form by

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ . \tag{1.3}$$

The inverse metric is then denoted by $\eta^{\mu\nu}$. We can also define the partial derivative operator as follows:

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) \ . \tag{1.4}$$

This is an operator, so for it to make sense it really needs to be acting on something. For example, if we have a scalar function ϕ that is a function of the spacetime coordinates x^μ, then $\partial_\mu\phi$ is a well-defined $(0, 1)$ tensor. Note that in all expressions above, the Greek indices μ and ν in all of these expressions range over the values $(0, 1, 2, 3)$, with each value corresponding to one of the four spacetime dimensions.

1.) We can define another derivative operator as follows:

$$\partial^\mu \equiv \eta^{\mu\nu}\partial_\nu \ . \tag{1.5}$$

Is this the same as ∂_μ? Write out $\partial^\mu\phi$ and $\partial_\mu\phi$ explicitly for some arbitrary function ϕ, and compare each component of these two vectors.

2.) Consider the following scalar function f in a four-dimensional Minkowski spacetime:

$$f(x^\mu) = atx + b(y^2 - xz) \ . \tag{1.6}$$

for some constants a and b. Then, write out the individual components of the vector field $\partial_\mu f$. Then, compute $(\partial_\mu f)(\partial^\mu f)$ as well as $\partial_\mu\partial^\mu f$. Are they the same?

3.) Consider a d-dimensional Minkowski spacetime with metric $\eta_{\mu\nu}$. In such a spacetime, we can define the Kronecker delta symbol δ^μ_ν as follows:

$$\delta^\mu_\nu = \begin{cases} 1 & \mu = \nu \\ 0 & \mu \neq \nu \end{cases} \ . \tag{1.7}$$

Prove that $\eta^{\mu\rho}\eta_{\nu\rho} = \delta^\mu_\nu$. Then, show that $\eta^{\mu\nu}\eta_{\mu\nu} = d$.

4.) Prove that, in a d-dimensional spacetime, $T^{\mu_1\cdots\mu_n} = \delta^{\mu_1}_\nu T^{\nu\mu_2\cdots\mu_n}$ for any arbitrary $(n, 0)$ tensor $T^{\mu_1\cdots\mu_n}$.

2 Manipulating the Einstein Equation

So far we have only talked about flat Minkowski spacetimes, which are simple because the metric tensor $\eta_{\mu\nu}$ is constant, and so the distance ds between any two points is the same all over the spacetime. However, as we will see in this class, general relativity allows for much more intricate spacetimes where the distance between two points is not in general constant. Instead, it is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \tag{2.1}$$

where we now have a more general metric tensor $g_{\mu\nu}$ that can actually be a function of the spacetime coordinates x^μ, instead of the Minkowski metric tensor $\eta_{\mu\nu}$. The inverse of the metric tensor is denoted by $g^{\mu\nu}$, and this inverse is defined such that

$$g^{\mu\rho} g_{\nu\rho} = \delta^\mu_\nu \ , \tag{2.2}$$

where δ^μ_ν is the Kronecker delta symbol defined in (1.10). Your result from problem 3 thus immediately tells you that the metric tensor and its inverse also satisfy the relation

$$g^{\mu\nu} g_{\mu\nu} = d \ , \tag{2.3}$$

where d is the number of spacetime dimensions. Additionally, the metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$ are still used to raise and lower indices, e.g.

$$A_\mu = g_{\mu\nu} A^\nu \ , \quad A^\mu = g^{\mu\nu} A_\nu \ . \tag{2.4}$$

The equations that govern the dynamics of the metric tensor are Einstein's equations, which are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = 8\pi G_N T_{\mu\nu} \ , \tag{2.5}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}$ and $T_{\mu\nu}$ are both $(0, 2)$ tensors, R is the trace of $R_{\mu\nu}$ (and so $R \equiv g_{\mu\nu} R^{\mu\nu}$), and Λ and G_N are both constants. We'll talk about this equation more in detail later in the semester, but for now we're going to get some practice working with it in order to make sure we understand index notation.

5.) Assume that we are in a d-dimensional spacetime, where $d > 2$. Show that the Einstein equation can be rewritten as

$$R_{\mu\nu} = 8\pi G_N T_{\mu\nu} + \frac{2}{2-d} g_{\mu\nu} \left(4\pi G_N T_\rho{}^\rho - \Lambda \right) \ . \tag{2.6}$$

6.) Now, assume that the spacetime is two-dimensional, i.e. $d = 2$. Show why the derivation in the previous problem no longer works. Then, show that the Einstein equation when $d = 2$ can instead be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\rho{}^\rho \right) \ . \tag{2.7}$$

7.) One particular class of spacetime metrics are *Einstein metrics*, for which the corresponding Ricci tensor is proportional to the metric:

$$R_{\mu\nu} = k g_{\mu\nu} \ , \tag{2.8}$$

for some constant k. Write Einstein's equations for such a metric, assuming an arbitrary number of spacetime dimensions d. Then, assume that $T_{\mu\nu} = 0$ and solve these equations for k.

3 Lorentz Transformations

Let's go back to four dimensions for this section. Lorentz transformations are the special class of coordinate transformations that leave the Minkowski metric invariant. That is, we consider coordinate transformations of the form

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \ . \tag{3.1}$$

In order for these coordinate transformations to preserve the infinitesimal distance ds between two points, they need to satisfy

$$\eta_{\rho\sigma} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \eta_{\mu\nu} \ . \tag{3.2}$$

This condition is also sometimes written in matrix form as $\eta = \Lambda^T \eta \Lambda$, though I think keeping track of the explicit indices is useful.

8.) Consider the following putative Lorentz transformation in matrix form:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \sqrt{1+a^2} & 0 & a & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & \sqrt{1+a^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{3.3}$$

where the rows correspond to the index μ and the columns to the index ν, and a is a constant. Prove that this is a Lorentz transformation.

9.) Write out $\Lambda_\mu{}^\nu$ explicitly in component form. As a check on your result, you should find that you do not get the exact same components as $\Lambda^\mu{}_\nu$.

10.) **Bonus:** Explain intuitively what this particular Lorentz transformation does by explicitly writing out how it acts as a coordinate transformation. Does it fall into the proper Lorentz group $SO(3,1)$, the restricted Lorentz group $SO(3,1)^\uparrow$, or neither? If neither, what's the issue?

11.) **Bonus:** Given two arbitrary Lorentz transformations $(\Lambda_1)^\mu{}_\nu$ and $(\Lambda_2)^\mu{}_\nu$, we can define their sum by $\Lambda^\mu{}_\nu \equiv (\Lambda_1)^\mu{}_\nu + (\Lambda_2)^\mu{}_\nu$. Is $\Lambda^\mu{}_\nu$ always guaranteed to be a Lorentz transformation? If so, prove it! If not, construct an explicit counterexample.

4 Actions and the Euler-Lagrange equations

In physics, many systems can be understood by writing down an *action*, denoted by S. Extremizing the action can then tell you something about how the system behaves.

In particular, let x^μ be our spacetime coordinates in a d-dimensional spacetime, and let $\Phi(x^\mu)$ be a field that takes a value at every point in our spacetime. The action for Φ can be expressed as

$$S[\Phi] = \int d^d x \, \mathcal{L}(\Phi, \partial_\mu \Phi, x^\mu) \ , \tag{4.1}$$

where $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ denotes a derivative with respect to the spacetime coordinates. The object $\mathcal{L}$ is called the *Lagrangian* if the system. $\mathcal{L}$ is a function of Φ as well as derivatives of Φ, which are in turn functions of the spacetime coordinates x^μ.

The action S is extremized when we find the true physical behavior of the field Φ. This extremum is determined by the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \Phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} \right) = 0 \ . \tag{4.2}$$

Note that when we take derivatives of $\mathcal{L}$ with respect to the field, we have to treat Φ and $\partial_\mu \Phi$ as independent objects, e.g. $\partial(\partial_\mu \Phi)/\partial \Phi = 0$.

12.) Let's start with a simple example from classical mechanics. Consider an object sitting in a constant gravitational field g, pointing downwards. We want to know what the height h of the ball above the ground is at all times t. In the language of our action, our coordinate is t, while the field is $h(t)$.

In classical mechanics, the Lagrangian is $\mathcal{L} = K - U$, where K is the kinetic energy and U the potential energy. The kinetic and potential energy for an object of mass m moving up and down in a constant gravitational field g are

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\partial_t h)^2 \ , \quad U = mgh \ , \tag{4.3}$$

and so the Lagrangian is

$$\mathcal{L} = \frac{1}{2}m(\partial_t h)^2 - mgh \ . \tag{4.4}$$

Use the Euler-Lagrange equation (4.2) to find a differential equation that determines the motion of the ball. Do you recognize this equation? (Hint: you should!)

Let's return to relativity and work within a four-dimensional spacetime with coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Let $\phi(x^\mu)$ be a free scalar field that lives on this four-dimensional spacetime. The Lagrangian for our field is given by

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 \ , \tag{4.5}$$

where m is the mass of the scalar field and ∂^μ is shorthand for $\partial^\mu = \eta^{\mu\nu}\partial_\nu = \eta^{\mu\nu}\frac{\partial}{\partial x^\nu}$. Note: it's okay if you've never seen something like this scalar field Lagrangian before. Intuitively, you should think of the first term in the Lagrangian as the "kinetic energy" of the scalar field, since it involves derivatives of the field, and the second term as the "potential energy" of the scalar field, since it involves the mass but no derivatives.

13.) First, show that

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2\phi \ . \tag{4.6}$$

14.) Next, we will need to differentiate the Lagrangian with respect to $\partial_\mu \phi$. But how do we differentiate something with respect to a vector field? The key in doing so lies in the following relation for any vector field A_μ:

$$\frac{\partial A_\nu}{\partial A_\mu} = \delta^\mu_\nu \ . \tag{4.7}$$

Try to motivate why equation (4.7) must be true. As a hint, try to think about the individual components of the vector field and what it means to differentiate something with respect to them.

15.) Use the general formula (4.7) from the previous problem to prove that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = -\partial^\mu \phi \ . \tag{4.8}$$

16.) Combine all your results so far to find the Euler-Lagrange equation (4.2) for the scalar field ϕ, and show that it can be written as

$$\partial_\mu \partial^\mu \phi - m^2\phi = 0 \ . \tag{4.9}$$

Then, write out this equation explicitly in terms of derivatives with respect to t, x, y, and z. Do you recognize what kind of equation this is?

17.) **Bonus:** Assume that ϕ is a function only of t, and not of x, y, or z. Solve the Euler-Lagrange equations for $\phi(t)$.